Period vs. Amplitude of a Simple Pendulum

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Abstract

We address the question of whether or not the period of a simple pendulum depends on it's amplitude. Experimental tests were done to compare the period of a simple pendulum for two different amplitudes. It was concluded from this experiment that the period of a pendulum does depend on it's amplitude. It was also found that the acceleration due to gravity in this lab was $g = 9.77 \pm 0.02\mathrm{m\,s^{-2}}$ and was found by knowing the period of the pendulum and the length of the pendulums arm.

1 Background

In this lab we investigate if the period of the pendulum is dependent upon the amplitude of the pendulum. This experiment also addresses whether or not it is possible to solve for the acceleration due to gravity by knowing the period and length of the pendulum. There is a mathematical equation for calculating the period of a pendulum that is tested in this lab for accuracy. As the period in this lab was calculated experimentally it was checked with numerical solutions using *Maple* to see if the approximation could be improved. It was later seen that their was an improved approximation for the period of a pendulum and is discussed later.

2 Theory

The differential equation that describes the motion of a simple pendulum is [1]

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta) \tag{1}$$

where θ is the amplitude of the pendulum, L is the length of the pendulum's arm, and g is the acceleration due to gravity that the pendulum is experiencing in equation (1). This experiment was conducted with small enough amplitudes that a simplification can be made to equation (1) with a small angle approximation ($\sin\theta \approx \theta$). With the substitution of $w_0 = \sqrt{g/L}$, equation (1) then becomes

$$\frac{d^2\theta}{dt^2} = -{\omega_0}^2\theta, \tag{2}$$

which looks very similar to equation (1). With the new variable substitution and small angle approximation equation (2) describes the motion of a simple pendulum. The period of a pendulum can be approximated by knowing other parameters (Length of pendulum arm and acceleration due to gravity) of the experiment.

2.1 Method of Experiment

Although we can measure the period of a pendulum by observing it oscillate we also want to be able to check what it should be with an equation. The period is defined to be $T = 2\pi/w_0$ where w_0 is the natural frequency of the oscillator [1]. Substituting for w_0 in the period equation, the equation that relates the period of the pendulum to the length of it's swinging arm is [1]

$$T = 2\pi\sqrt{\frac{L}{g}}, \tag{3}$$

giving a mathematical approximation for calculating the period of the pendulum. The L is the length of the pendulum arm and the g is the acceleration due to gravity in equation (3). Once a value for the period of a pendulum is known the acceleration due to gravity can be calculated. Since equation (3) solves for the period of the pendulum with gravity in the equation, it can be rearranged so that it is determining the acceleration due to gravity in terms of the arm length and period of the pendulum. Doing some algebra the following rearranged equation is

$$g = \frac{4\pi^2 L}{T^2}, \tag{4}$$

showing how the acceleration due to gravity can be calculated with these other known values. By physically measuring the length of the pendulum arm we can find the arm length L and by doing the experiment we can find the period T. From this information through physical measurements and experimentation, equation (4) is then used to find the acceleration due to gravity with this experiments data. Once these equations were known, the experiment started with taking the measurements of the wire with the hanging ball suspended from it. A reference point was then made on the table that was under where the ball was being suspended (This is where the periods were counted when the pendulum was swinging). Then the suspended ball was pulled back and released, for two different amplitudes ten times each where the timing for the pendulum was started at the minimum of it's trajectory. Once the pendulum completed ten oscillations the time was stopped and the period was calculated from knowing this information. From knowing the displacement of the pendulum and it's arm length the amplitude of the oscillation was calculated. Once the period of the pendulum was then known the acceleration due to gravity was then calculated. Table 1 and Table 2 have the data that came from the data runs previously mentioned.

2.2 Period Data

Run	Total Time (s)	T (s)	$\triangle x(cm)$
1	24.78	2.48	10
2	24.83	2.48	10
3	24.79	2.48	10
4	24.84	2.48	10
5	24.87	2.49	10
6	24.87	2.49	10
7	24.86	2.49	10
8	24.88	2.49	10
9	25.16	2.52	10
10	24.99	25.0	10

Table 1

Table 1 specifically is the data that came from the smaller amplitude run of this experiment. Data for the larger amplitude is in Table 2.

Run	Total Time (s)	T (s)	$\triangle x(cm)$
1	24.86	2.49	30
2	24.81	2.48	30
3	24.87	2.49	30
4	24.91	2.49	30
5	24.78	2.48	30
6	24.86	2.49	30
7	24.84	2.48	30
8	24.85	2.49	30
9	24.93	2.49	30
10	24.82	2.48	30

Table 2

Data from Table 1 gives an average period of $T = 2.489$s, a standard deviation of 0.0106, and a standard deviation of the mean of 0.0034. Data from Table 2 gives an average period of $T = 2.485$s, a standard deviation of 0.0042, and standard deviation of the mean of 0.0013. The period of the smaller amplitude run can then be stated as

$$T_{Smaller} = 2.489 \pm 0.003\text{s}, \tag{5}$$

and the period for the larger amplitude is then stated as

$$T_{Larger} = 2.4853 \pm 0.0013\text{s} \tag{6}$$

with results (5) and (6) giving the two periods of the two different displacements. The ranges of the periods from result (5) and (6) overlap indicating that there is no measureable difference between the two.

2.3 Amplitude Data

The pendulum's trigonometry needs to be discussed for how the amplitudes of the experiment trials were calculated. Figure 1 shows how the trigonometry of this experiment looked visually. The $\triangle x$ value is the displacement horizontally of the pendulum, the L is the length of the pendulum arm, and the θ is the amplitude of the pendulum.

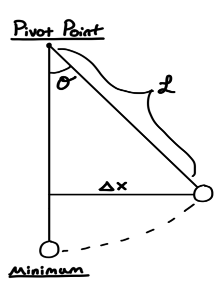

Figure 1

The two horizontal displacements and the length of the pendulum arm were all the measurements that were needed to get the amplitude of these runs. The length of the pendulum arm came out to be $L = 1.532 \pm 0.001$m for this experiments set up and did not change between the smaller and larger amplitude runs. The equation that defines how the amplitude was calculated is

$$\theta = \arcsin\left(\frac{\triangle x}{L}\right), \tag{7}$$

and was derived from the trigonometry in Figure 1. Equation (7) gives a value of $\theta = 0.0653(rads)$ for the smaller amplitude run while equation (7) gives a value of $\theta = 0.1971(rads)$ for the larger amplitude run. The following table shows the average period with its error and amplitude of each trial set.

Size	T (s)	θ (radians)
Smaller	2.489 ± 0.003	0.0653
Larger	2.4853 ± 0.0013	0.1971

Table 3

While the amplitude raises by a factor of three from the smaller to the bigger amplitude, the period of the two remain to be very similar.

2.4 Solving For g

Since the period and amplitude of both magnitudes have been calculated the attention can shift to solving for the acceleration due to gravity. The values in equation (1) that *Maple* used to solve equation (1) numerically were $L = 1.532$m and $g = 9.79\mathrm{m\,s^{-2}}$. Using the famous gravitational force equation of $F = GMm/r^2$ [2] for where this experiment happened, the acceleration due to gravity at 1.4 km is $g = 9.79\mathrm{m\,s^{-2}}$. This is why the value for g is not the typical value of $g = 9.80\mathrm{m\,s^{-2}}$. Since this experiment involved two different initial release points with both being released from rest, there will be two periods that *Maple* finds solutions for. For the smaller initial displacement of $\theta = 0.0653(rads)$ with the use of maple's differential equation solver the period was found to be $T = 2.488$s. This was done by first numerically solving equation (1) with the aide of *Maple*. Once equation (1) was solved, it was plotted against time and it was observed on the graph by zooming in and out where one full period was by finding where the plot intercepted the time axis. For the larger amplitude the initial displacement was $\theta = 0.1971(rads)$ and the period came out to be $T = 2.493$s according to *Maple*. The same method of looking at the plot of the solution and then finding where the solution crossed the axis for one period was used for solving for the period of the larger amplitude run. These results for the periods from *Maple* are slightly different from the ones that were found by experimentation. The analytical method (Experimental method) was with the timing of the oscillations for the period of the pendulum and finding the averages among a certain data set. The numerical was with the aide of *Maple*. The reason for this discrepancy in values is discussed shortly.

The discrepancies between the analytical and numerical solutions arise from a correction term that is added to equation (3). The values found previously with *Maple* can be found with this new correction term. The new equation for timing the period of a pendulum from the length of the pendulum arm with the acceleration due to gravity looks like [3]

$$T = 2\pi\sqrt{\frac{L}{g}}(\frac{\ln(a)}{1-a}), \tag{8}$$

where equation (8) has an algebraic substitution for a of

$$a = \cos(\frac{\theta}{2}), \tag{9}$$

now giving a more precise equation for the period of a simple pendulum. The a variable in equation (9) helps account for the amplitude contribution to the period of a pendulum, whereas the θ variable is the amplitude of the displacement of the pendulum with units in radians. With this new knowledge of equation (8) and (9), a refined equation (4) can be constructed. Rearranging and doing some algebra equation (4) turns into

$$g = \frac{4\pi^2 L}{T^2}(\frac{\ln(a)}{1-a})^2, \tag{10}$$

yielding a more accurate equation for the acceleration of gravity. Using the data from the experiment of $T = 2.489$s, $L = 1.532$m, and $\theta = 0.0653(rads)$, the smaller amplitude run had an acceleration due to gravity of $g = 9.77\mathrm{m\,s^{-2}}$ when using equation (10) for the calculation. When using the data of $T = 2.4853\mathrm{m\,s^{-2}}$, $L = 1.532$m, and $\theta = 0.1971(rads)$ for the larger amplitude run, the acceleration due to gravity came out to be $9.84\mathrm{m\,s^{-2}}$ when using equation (10). With a calculated error propagation of $\pm 0.02\mathrm{m\,s^{-2}}$, the following table is constructed to summarize the results from the observed periods for what the acceleration due to gravity should be.

θ (radians)	$T(s)$	$g(\frac{m}{s^2})$
0.0653	2.489 ± 0.003	9.77 ± 0.02
0.1971	2.4853 ± 0.0013	9.84 ± 0.02

Table 4

Equation (10) was the equation used to find the acceleration due to gravity from the observed period in this experiment. Although the values for the acceleration due to gravity are different when they are calculated, this should not happen in theory since the different runs for the experiment happened at the same elevation. The smaller amplitude run is the only value that is acceptable with reference to the correct acceleration due to gravity. The larger amplitude run has errors in it that came from human error while observing the period of the oscillation. The accepted acceleration due to gravity was found to be $g = 9.79\mathrm{m\,s^{-2}}$ and is why the only acceptable g value is from the smaller amplitude run due to it being in the range of the accepted value. Equation (8) provides a more precise equation for calculating the period of a pendulum when the amplitude, the length of the arm, and acceleration due to gravity are known for the experiment. The *Maple* found solutions for the period of the pendulum when the acceleration due to gravity is $g = 9.79\mathrm{m\,s^{-2}}$ are in Table 5.

θ (radians)	$T(s)$
0.0653	2.489
0.1971	2.493

Table 5

This time when *Maple* was used to solve for the periods numerically the larger amplitudes period increased from what was previously seen in Table 4. This means that the discrepancy between the analytical periods and the numerical periods come from the correction term introduced in equations (8), (9), and (10). This means that the correction term is a better approximation for calculating the period of a pendulum as well as the acceleration due to gravity from that pendulum. This improvement in calculation is due to an incorporation of the amplitude in equations (8) and (10).

3 Discussion of Results

3.1 Data

When the correction term was added to equation (10) it was used to calculate what the acceleration due to gravity should be from the observed periods. These values of acceleration due to gravity were then referenced to see if they were accurate. It was then confirmed that the smaller amplitude run had an acceptable acceleration due to gravity of $g = 9.77 \pm 0.02\mathrm{m\,s^{-2}}$ which was in the range for the accepted value. When the data is ran through *Maple* to calculate the period and acceleration due to gravity, the periods are found to be different from one another but the acceleration due to gravity in theory should not be different since the separate runs were done at the same elevation. Since the acceleration due to gravity doesn't change in this experiment, but the periods of the runs do, there is another contributing factor to the period. Since the only other term in equation (8) as well as (10) is the length of the pendulum arm (And this doesn't change in the experiment) we can conclude that the difference in period comes from the amplitude of the pendulum because of it's introduction into the more accurate equations of (8) and (10). This contribution is the a term seen in equations (8) and (10) and defined in equation (9). Both ways of calculating the acceleration due to gravity and the period of the pendulum help address the problem initially stated for this experiment.

3.2 Conclusion

From the data that was originally collected it was observed that the period of a pendulum does not depend on it's amplitude. When the correction term for the period was introduced in equation (8), discrepancies between the periods of the runs increased when a more accurate value was calculated. These increase in discrepancies between periods suggest that there is another contributing factor towards calculating the period of a pendulum And when *Maple* was used to solve for the period of the pendulum with the known acceleration due to gravity it was observed that the period of a pendulum depended upon more than just gravity and the length of its arm. It can then be concluded because of this reason that the period of a pendulum does depend on its amplitude. Observations for this were made in the experiment by seeing a different period while it was proved with the use of *Maple* and equations (8), (9), and (10).

References

[1] King, George C. Vibrations and Waves. Vol. 1. Chichester: Wiley, 2009. 1 vols.

[2] Knight, R. D. (2012). Physics for Scientists and Engineers a Strategic Approach. Pearson. Retrieved March 3, 2018

[3] Lima, F.M.S. "An Accurate Formula for the Period of a Simple Pendulum Oscillating Beyond the Small Angle Regime." American Journal of Physics 74 (2006):